Geodesic Motions Versus Hydrodynamic Flows in a Gravitating Perfect Fluid: Dynamical Equivalence and Consequences

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Abstract

Stimulated by the methods applied for the observational determination of masses in the central regions of the AGNs, we examine the conditions under which, in the interior of a gravitating perfect fluid source, the geodesic motions and the general relativistic hydrodynamic flows are **dynamically equivalent** to each other. Dynamical equivalence rests on the functional similarity between the corresponding (covariantly expressed) differential equations of motion and is obtained by conformal transformations. In this case, the spaces of the solutions of these two kinds of motion are **isomorphic**. In other words, given a solution to the problem *hydrodynamic flow in a gravitating perfect fluid*, one can always construct a solution formally equivalent to the problem *geodesic motion of a fluid element* and vice versa. Accordingly, we show that, the *observationally determined nuclear mass* of the AGNs is being **overestimated** with respect to the real, physical one. We evaluate the corresponding **mass-excess** and show that it is not always negligible with respect to the mass of the central dark object, while, under circumstances, can be even larger than the rest-mass of the circumnuclear gas involved.

1 Introduction

It is generally believed that active galaxies are powered by the presence of central, massive black holes (Rees 1984; Blandford & Rees 1992). Many galaxies may have gone through an active phase and, therefore, massive black holes could be common in any *active galactic nucleus* (AGN) phenomenon (Holt et al. 1992; Antonucci 1993; Urry & Padovani 1995). Moreover, black holes are expected to be present in the

centers of many quiescent galaxies as well (Chokshi & Turner 1992). However, direct dynamical evidence for the presence of black holes in the central regions of individual galaxies is scarce. Stellar kinematical studies have provided tentative evidence for black holes only in a handfull of nearby galaxies, because of difficulties in spatial resolution and the lack of knowledge on the exact shape of the stellar orbits (Kormendy & Richstone 1995; Ho 1998). Based on stellar dynamics, the mass concentrations in the central regions of these galaxies are determined through Doppler-shift measurements involving Keplerian motions. Accordingly, the velocity dispersion $\upsilon(r)$ of various sources of radiation located at a distance r from the center, is usually assumed to follow the standard virial-theorem equation for circular geodesic motion (Peebles 1972; Quinlan et al. 1995)

$$\upsilon^2(r) = G\,\frac{M(r)}{r} \tag{1}$$

where $M(r)$ is the corresponding distribution of mass inside the radius r and G is the Newton's gravitational constant. In fact, it was this law which led Sargent et al. (1978) to estimate, for the first time, the mass of the central black hole in M87, to be of the order $5 \times 10^9\ M_\odot$.

On the other hand, recent observational data indicate that, in most of the AGNs, there exist gas clouds surrounding the central dark object and the associated *accretion disk*, on a variety of scales from a tenth of a parsec to a few hundreds parsecs (see e.g. Holt et al. 1992; Urry & Padovani 1995). These clouds are assumed to form a geometrically and optically thick torus (or warped disc), which absorbs much optical UV radiation and soft X-rays (Guilbert & Rees 1988; Rees 1995). Since 1993, many indications appeared in favour of the above idea, e.g. by HST imaging of gas and dust in NGC 4261 (Jaffe et al. 1993), by the mapping in nearby AGNs of the thermal molecular CO emission in Centaurs A (Rydbeck et al. 1993), or by the high-density tracer HCN in NGC 1068 (Tacconi et al. 1994) and M51 (Kohno et al. 1996). This *circumnuclear gas* seems to exist in either the molecular or the atomic phase, the latter being due to hard X-ray sources in the centers of most radio-loud AGNs (Krolik & Begelman 1988; Maloney et al. 1994). In fact, to date the most dramatic evidence for the existence of a supermassive black hole comes from the VLBI imaging of molecular (H_2O) *masers* in the active galaxy NGC 4258 (Greenhill et al. 1995; Miyoshi et al. 1995). This imaging, produced by Doppler-shift measurements assuming Keplerian motion of the masering source, has allowed a quite accurate estimate of the nuclear central mass, which has been found to be a $3.6 \times 10^7\ M_\odot$ supermassive dark object, within 0.13 parsecs. **However, this evidence (as well as many of the previous ones) comes primarily from the motions of gas streams rather than stars**. The gas is also subject to non-gravitational forces and may not follow ballistic trajectories. Indeed, it has been recently announced (Greenhill & Gwinn 1997) that the rotation curve of the active galaxy NGC 1068 (in which, maser emission has also been detected) is sub-Keplerian, suggesting that a non-gravitational component is also present in the maser emission. Therefore, as regards the measurement data relevant to the cental dark objects in the AGNs, there is some inherent ambiguity

(Rees 1995).

In fact, the determination of the central nuclear mass is based on the Doppler-shifted radiation received, not from something like a *test-particle* star, but from the *extented* masering fluid sources. These sources move in the gravitational field of a (more or less) continuous gravitating source, such as the circumnuclear gas and the accretion disk, and from a dynamical point of view, the assumption that the motion of the masering source is Keplerian (or, in general, *geodesic*), seems to be *inaccurate*. On the other hand, a fluid *volume element* can be considered as more realistic constituent of a fluid source and hence, its use in a continuous medium should be physically preferable than that of a theoretical, ideal test-particle (Spyrou 1997a; 1997b; 1997c). So, the question is raised as to *whether the hydrodynamical flow of a fluid volume element in a continuous gravitating source can be approximated or not to the motion of a test-particle (namely the geodesic motion) in the same source.*

Motivated by the above considerations, in the present article we examine the conditions under which the hydrodynamic flows in the interior of a bounded self-gravitating perfect fluid source may become **dynamically equivalent** to general-relativistic geodesic motions in this source. It is clear that, generally, in the interior of a continuous perfect fluid source the hydrodynamic flow differs from the geodesic motion. In this case, dynamical equivalence rests in the fact of the functional similarity between the corresponding (covariantly expressed) differential equations of motion (Spyrou 1999). Accordingly, the spaces of the solutions of these equations are *isomorphic*. In other words, given a solution to a problem of *hydrodynamic flow in a gravitating perfect fluid*, one can always construct a solution formally equivalent to the problem *geodesic motion of a fluid element* and vice versa. The problem attacked is kept as general as possible, being solved within the context of the exact General Relativity Theory (GRT). Therefore, the corresponding results could be useful in many astrophysical situations and not just in the context of relativistic galactic dynamics and/or cosmology.

2 Geodesic Motions Versus Hydrodynamic Flows

The ballistic motion of a test-particle in the gravitational field of a bounded, self-gravitating perfect fluid source, takes place along *geodesic* trajectories, on which the tangent vector of the four-velocity $u^i = \frac{dx^i}{ds}$ $(u_i u^i = 1)$ obeys the standard *geodesic equation* in a coordinate frame x^i (see e.g. Papapetrou 1974)

$$\frac{Du^i}{ds} = 0 \quad \Leftrightarrow \quad \frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = 0 \tag{2}$$

where the Latin indices, denoting spacetime coordinates, admit the values $0, 1, 2, 3$ and the Greek indices, denoting spatial coordinates, admit the values $1, 2, 3$. In Eq. (2), $\frac{Du^i}{ds}$ denotes the *covariant acceleration* vector, s is the *affine parameter* and Γ^i_{kl} are the *Christoffel symbols*. On the other hand, the equations of motion of the hydrodynamic

flow in the interior of the same source, are written in form

$$\mathcal{T}^{ij}_{\ ;\,j} = 0 \tag{3}$$

where a semi-colon denotes covariant differentiation and $\mathcal{T}^{ij}$ is the stress-energy tensor of the perfect fluid, which is written in the standard form

$$\mathcal{T}^{ij} = (\mathcal{E} + p)u^i u^j - p g^{ij} \tag{4}$$

In Eq. (4), p is the isotropic pressure and $\mathcal{E}$ is the overall mass-energy density of the perfect fluid. Both quantities are considered to transform like scalars under coordinate mappings (Anderson 1967). In the absence of *shear* and *viscocity*, $\mathcal{E}$ is decomposed as follows (Fock 1959; Chandrasekhar 1965)

$$\mathcal{E} = \rho c^2 + \rho \Pi \tag{5}$$

where, $\rho c^2 \neq 0$ is the rest-mass energy density of the total number of baryons included in the unit volume element, and $\rho\Pi$ is the corresponding specific internal energy density, the variations of which are related to the compressions or the expansions of the fluid. Combination of Eqs. (3) and (4) yields

$$(\mathcal{E} + p)_{\,,\,j}\, u^i\, u^j + (\mathcal{E} + p)\,(u^i_{\,;j}\, u^j + u^i\, u^j_{\,;\,j}) - p_{\,,\,j}\, g^{ij} = 0 \tag{6}$$

Now, in order to study a possible dynamical equivalence of the hydrodynamic flow of a finite-volume element in the interior of a bounded, self-gravitating perfect fluid to the corresponding geodesic motions, first of all we need to determine the evolution of the fluid's covariant acceleration. In this case, it is convenient to study the flow motion on a hypersurface normal to the direction of u^i. We do so, by contracting Eq. (6) with the *projection operator*

$$h_{ik} = g_{ik} - u_i u_k \tag{7}$$

for which, we obviously have

$$h_{ik} u^i = 0 \tag{8}$$

Then, provided that $\mathcal{E} + p \neq 0$, i.e. the total *enthalpy* of a fluid element is non-vanishing, the combination of Eqs. (6) and (7) yields

$$u_{k;\,j} u^j = \frac{p_{,\,j}}{(\mathcal{E} + p)} \left(\delta^j_k - u_k u^j \right) \tag{9}$$

Eq. (9) is just one of the two equations of fluid motion, in which Eq. (3) splits in the case of a perfect fluid (Taub 1959). This equation is completely analogous to the *conservation-of-momentum* equation of Newtonian hydrodynamics (Ryan & Shepley 1975). In a Riemannian space ($g_{ij;s} = 0$), Eq. (9) is written in the form

$$g_{kl} u^l_{\,;\,j} u^j = \frac{p_{,\,j}}{(\mathcal{E} + p)} \left(\delta^j_k - u_k u^j \right) \tag{10}$$

from which, on contracting with g^{mk}, we obtain

$$\delta_l^m \left[\frac{du^l}{ds} + \Gamma^l_{js} u^j u^s \right] = g^{mk} \frac{p_{,j}}{(\mathcal{E} + p)} \left(\delta_k^j - u_k u^j \right)$$

thus, resulting to

$$\frac{du^m}{ds} + \Gamma^m_{js} u^j u^s = g^{mk} \frac{p_{,j}}{(\mathcal{E} + p)} \left(\delta_k^j - u_k u^j \right) \tag{11}$$

or, finally,

$$\frac{du^m}{ds} + \Gamma^m_{js} u^j u^s = \frac{1}{\mathcal{E} + p} \, h^{mj} \, p_{,j} \tag{12}$$

From Eq. (12), it becomes evident that, the two kinds of motion are completely (not just dynamically) equivalent, provided that the pressure gradient in the three-surface normal to u^i [which appears in rhs of Eq. (12)] is zero. This quantity measures the response of a particle to non-gravitational fields and, therefore, it is responsible for all deviations of flow lines from geodesics (see also Misner et al. 1973; Ryan & Shepley 1975).

3 Dynamical Equivalence Conditions

By definition, dynamical equivalence between geodesic motions and hydrodynamic flows rests in the functional similarity between Eqs. (2) and (12). In this case, the spaces of the solutions of these equations are isomorphic. In other words, given a solution to a problem of *hydrodynamic flow in a perfect fluid*, one can always construct a solution formally equivalent to the problem *geodesic motion of a fluid element* and vice versa. If we manage to establish dynamical equivalence between Eqs. (2) and (12), then we will be able to extrapolate every result obtained on the basis of geodesic motions, to the more realistic context of general-relativistic fluid hydrodynamics. A direct application may concern observational data from many relativistic astrophysical (or cosmological) systems, such as the central cores of the AGNs (Kormendy & Richstone 1995; Rees 1995) or the existence and the nature of the *dark matter* (Saglia 1996) etc. Dynamical equivalence between Eqs. (2) and (12) may be performed in two ways:

1. By eliminating the rhs of Eq. (12). A direct way to do so is to assume *isobaric motions*, i.e. $p = constant$. However, in general, this is not always a physically necessary condition.

2. By transfering the problem to a *virtual* self-gravitating perfect fluid, the metric of which is *conformal* to g_{kl}. In terms of this fluid, Eq. (12) will be subsequently written in the form of Eq. (2).

The latter, is a very delicate method of obtaining dynamically equivalent descriptions in any gravitational field theory and not just GRT (e.g. see Spyrou 1976; Whitt 1984; Cotsakis 1993). A conformal transformation (Penrose 1964; Hawking & Ellis 1973) shrinks or stretches the entire manifold, introducing a change in the scale of all lengths and time, which can be different for the various world points, but is the same for all spatial directions and time at a given point. A conformal transformation of a metric is usually described by

$$\tilde{g}_{kl} = \Omega^2(x^i)\, g_{kl} \tag{13}$$

for some continuous, non-vanishing, finite, real function $\Omega(x^i)$. Accordingly, in what follows, we assume that there exists a virtual gravitating perfect fluid, producing a **new** metric tensor $(\tilde{g}_{kl})$, in terms of which we may write Eq. (12) in the form of Eq. (2), namely

$$\frac{d\tilde{u}^m}{d\tilde{s}} + \tilde{\Gamma}^m_{kl}\tilde{u}^k\tilde{u}^l = 0. \tag{14}$$

Next, we shall try to determine the relation between these two fluids. To begin with, we note that when Eq. (13) is applied, the following transformations hold (Hawking & Ellis 1973)

$$\begin{aligned} \tilde{\Gamma}^m_{kl} &= \Gamma^m_{kl} + \frac{1}{\Omega}\,(\delta^m_k \Omega_{,\,l} + \delta^m_l \Omega_{,\,k} - g_{kl} g^{ms} \Omega_{,\,s}) \\ \frac{d\tilde{u}^m}{d\tilde{s}} &= \frac{1}{\Omega}\,\frac{\partial}{\partial x^n}\left(\frac{1}{\Omega}\right) u^m u^n + \frac{1}{\Omega^2}\,\frac{du^m}{ds} \end{aligned} \tag{15}$$

With the aid of Eqs. (15), the geodesic equation (14) may be decomposed in terms of the original metric tensor (g_{kl}), as follows

$$\frac{du^m}{ds} + \Gamma^m_{kl}u^k u^l = h^{mn}\,\frac{\partial}{\partial x^n}(ln\Omega) \tag{16}$$

Now, in view of the above definitions, dynamical equivalence between the two kinds of motion under consideration, implies

$$h^{mn}\,\frac{\partial}{\partial x^n}(ln\Omega) = \frac{1}{\mathcal{E}+p}\,h^{mn}\,p_{,\,n} \tag{17}$$

To solve Eq. (17) with respect to Ω is a very difficult task, especially when no further conditions are imposed. Nevertheless, there exists a quite general physical condition which may help us to reach at some (at least) particular solutions. It is the *equilibrium hydrodynamics hypothesis*, i.e. the constancy of *entropy* $(\mathcal{S})$ along the original fluid's flow lines (Taub 1959; Misner et al. 1973)

$$\mathcal{S}_{,\,m}\,u^m = 0 \tag{18}$$

In fact, in relativistic astrophysics and cosmology we usually assume $\mathcal{S}$ to be a group invariant (constant in space) and therefore $\mathcal{S} = const.$ (Ryan & Shepley 1975). Hence, in what follows we consider that the flow motion is *isentropic*.

Adiabaticity of the hydrodynamical motions could be physically necessary, because then both the thermodynamic and the matter content (the number of baryons) of a finite-volume element of perfect fluid remain constant during its motion (Chandrasekhar 1983). Then, the general-relativistic analogue of the *first law of thermodynamics* is valid with respect to the original fluid, for $d\mathcal{S} = 0$. In the case of a perfect fluid source, this law may be expressed in the form

$$\Pi_{,\,m} + p \left(\frac{1}{\rho}\right)_{,\,m} = 0 \tag{19}$$

or, more conveniently,

$$p_{,\,m} - \rho c^2 \left(\frac{\mathcal{E} + p}{\rho c^2}\right)_{,\,m} = 0 \tag{20}$$

Inserting Eq. (20) into Eq. (17), we obtain

$$h^{mn} \frac{\partial}{\partial x^n} \left[ln\,(\Omega) \; - \; ln \left(\frac{\mathcal{E} + p}{\rho c^2}\right)\right] = 0 \tag{21}$$

In Eq. (21), we perform the substitution

$$\Omega(x^k) \; = \; \frac{\mathcal{E} + p}{\rho c^2}\, e^{\Phi(x^k)} \tag{22}$$

Then, the new metric ($\tilde{g}_{mn}$), in terms of the original one (g_{mn}), is written in the form

$$\tilde{g}_{mn} \; = \; \left(\frac{\mathcal{E} + p}{\rho c^2}\right)^2 e^{2\Phi(x^k)}\, g_{mn} \tag{23}$$

and the dimensionless, real scalar function $\Phi(x^k)$ satisfies the *compatibility condition*

$$h^{mn}\, \Phi_{,\,n} = 0 \tag{24}$$

Eq. (24) has a clear physical meaning: The projection of the vector $\Phi_{,\,n}$ on a three-dimensional hypersurface normal to u_n is zero. By virtue of Eq. (8), this condition implies that $\Phi_{,\,n}$ is always *collinear* to u_n. Accordingly, the **general solution** to Eq. (24) may be obtained in terms of the differential equation

$$\Phi_{,\,n} = \zeta(x^m)\, u_n \tag{25}$$

(in connection see Anderson 1967; Ryan & Shepley 1975) where $\zeta(x^m)$ is an arbitrary scalar function with dimensions of inverse length. The general solution to Eq. (25), may be expressed in the form

$$\Phi(x^m) = \int \zeta(x^m)\, u_n\, dx^n + \xi(x^r) \tag{26}$$

where no summation is assumed over n, and $\xi(x^r)$ is a dimensionless scalar function of the coordinates x^r $(r \neq n)$. Obviously, the exact form of the general solution to Eq. (24) remains undetermined. This is not an unexpected result, since Eq. (24) is a system of four partial differential equations involving five unknown quantities, namely, the four components of $u^i(x^n)$ and $\Phi(x^n)$. In this case, supplementary conditions should be imposed, for someone to deal with a well-defined problem.

Nevertheless, there exists one **particular solution** to Eq. (24) which is of great theoretical interest, namely

$$\Phi(x^n) = \mathcal{C} = const. \tag{27}$$

In this case, the combination of Eqs. (23) and (27) relates the physical metric tensor g_{mn}, due to the gravitational field in the interior of a real perfect fluid source (in which the motions follow the laws of relativistic hydrodynamics), to the correponding metric $\tilde{g}_{mn}$ of a virtual fluid (in which the original hydrodynamic motions become geodesic). According to Eq. (23), the functional form of the metric attributed to the new gravitational source includes the overall enthalpy content $(\mathcal{E}+p)$ within a *specific volume* $(\frac{1}{\rho})$ of the original one (*specific enthalpy*).

The family of conformal transformations (13) [or (23)] represents an *algebraic group* over the space of the Riemannian metrics. In this case, it is evident that the corresponding *operation* among the various group elements is the standard multiplication between the real, finite, non-zero, scalar functions $\Omega(x^n)$. There exists an *identity* element $(\Omega = 1)$, as well as an *inverse* one (Ω^{-1}), such that $\Omega^{-1} \cdot \Omega = 1$. As regards the identity element, it represents the degenerate case, in which the associated metrics (the real and the virtual) are **identical**, $\tilde{g}_{mn} = g_{mn}$. In our problem, this corresponds to an *isobaric flow* of the original fluid [e.g. see Eq. (12)]. For $p = p_0 = const.$, the first law of thermodynamics [Eq. (19)] is reduced to

$$\Pi + \frac{p_0}{\rho} = const. \tag{28}$$

and the identity transformation, through Eqs. (23) and (27), implies

$$\mathcal{C} = -\, ln \left[1 + \frac{1}{c^2}(\Pi + \frac{p_0}{\rho}) \right] = const. \tag{29}$$

Therefore, if the adiabatic flow of a finite-volume element in the interior of a bounded, self-gravitating perfect fluid is (in addition) isobaric, then the corresponding metric tensor is **conformally invariant** under the group of transformations (23), and the hydrodynamic flows are completely equivalent to geodesic motions.

In what follows, we shall assume the validity of Eq. (27) and, unless otherwise stated, we shall ignore the constant factor $e^{2\mathcal{C}}$ arising in the formula for Ω from the combination of Eqs. (23) and (27). This normalization may be performed through a redefinition of the proper length in the original metric, as $s \rightarrow \lambda = s\, e^{-\mathcal{C}}$. In concluding, the appropriate conformal transformation which guarantees the dynamical equivalence between hydrodynamic flows and geodesic motions in a gravitating

perfect fluid, is

$$\tilde{g}_{mn} = \left(\frac{\mathcal{E} + p}{\rho c^2} \right)^2 g_{mn} \tag{30}$$

provided that the following major assumptions are valid:

1. We have considered a gravitating perfect fluid, i.e. isotropic in pressure, with no shear and viscocity.
2. In this source, the hydrodynamic flow motions are adiabatic, i.e. $\mathcal{S}$ is constant along the fluid flow lines.
3. The gravitational field in the interior of this fluid corresponds to a Riemannian spacetime, i.e. $g_{\,kl\,;\,m} = 0$ and $det|g_{kl}| \neq 0$.
4. The corresponding metric tensor is a solution to the Einstein field equations, through $\mathcal{T}^{kl}_{\,;\,l} = 0$.

4 Reduction to the Newtonian Limit

In many cases of particular astrophysical interest, the corresponding *weak field-limit* results seem to be more appropriate. For example, it has already been shown (Spyrou & Varvoglis 1982) that, in the context of relativistic galactic dynamics and astrophysics, the use of the first post-Newtonian approximation is more than enough in most of the cases. In fact, for astrophysically interesting sources with non-negligible relativistic characteristics (such as e.g. the nuclei of the giant elliptical galaxies) the small post-Newtonian expansion parameter (for the solar system being $\approx 10^{-6}$) is still very small ($\approx 10^{-3} \ll 1$). Accordingly, we will examine how the above results, obtained within the context of the exact GRT, may affect on the corresponding Newtonian limit. In this limit, as regards the metric tensor of the original gravitating perfect fluid, we have (Schiff 1960; Chandrasekhar 1965)

$$\begin{aligned} g_{00} &= 1 - 2\frac{U}{c^2} + O_4 \\ g_{0\alpha} &= O_3 \\ g_{\alpha\beta} &= -\delta_{\alpha\beta} + O_2 \end{aligned} \tag{31}$$

where the symbol O_n denotes terms of order n in the small parameter $\frac{\varepsilon}{c} \ll 1$, with

$$\varepsilon^2 = max\,(v^2, U, \Pi, \frac{p}{\rho}) \tag{32}$$

with v denoting the measure of the spatial three-velocity vector and U the gravitational potential, related to the mass density ρ through the standard Poisson equation

$$\nabla^2 U = -4\pi G\,\rho \tag{33}$$

According to Eq. (30), the components of the metric tensor attributed to the virtual perfect fluid, may be written in terms of the corresponding components of the original one (31), as follows

$$
\begin{aligned}
\tilde{g}_{00} &= \left(\frac{\mathcal{E}+p}{\rho c^2}\right)^2 \left(1 - 2\frac{U}{c^2} + O_4\right) \\
&= \left(1 + 2\,\frac{1}{c^2}\,\left[\Pi + \frac{p}{\rho}\right] + O_4\right)\,\left(1 - 2\frac{U}{c^2} + O_4\right) \\
&= 1 - 2\,\frac{\tilde{U}}{c^2} + O_4
\end{aligned}
\tag{34}
$$

In this case, from Eq. (34) we observe that, $\tilde{g}_{00}$ is formally equivalent to g_{00}, provided that U is replaced by the *virtual gravitational potential*

$$\tilde{U} = U - \left(\Pi + \frac{p}{\rho}\right) \tag{35}$$

Eq. (35) is identical to the corresponding Newtonian result of Spyrou (1997a; 1997b; 1997c). In the same fashion, we obtain

$$\tilde{g}_{0\alpha} = O_3 = g_{0\alpha} \tag{36}$$

and

$$
\begin{aligned}
\tilde{g}_{\alpha\beta} &= (-\delta_{\alpha\beta} + O_2)\,\left[1 + 2\,\frac{1}{c^2}\,\left(\Pi + \frac{p}{\rho}\right) + O_4\right] \\
&= -\delta_{\alpha\beta} + O_2 \\
&= g_{\alpha\beta}
\end{aligned}
\tag{37}
$$

The set of Eqs. (34) - (37) represents the appropriate weak field-limit transformation, relating the conformal metrics of the real and the virtual perfect fluid, under the action of the group (30). In analogy to Eq. (33), we may define the *virtual rest-mass density* ($\tilde{\rho}$), which produces the corresponding gravitational potential $\tilde{U}$, through

$$\nabla^2 \tilde{U} = -4\pi G\,\tilde{\rho} \tag{38}$$

which, in view of Eq. (35), yields

$$\tilde{\rho} = \rho + \frac{1}{4\pi G}\,\nabla^2(\Pi + \frac{p}{\rho}). \tag{39}$$

5 The Conformal Stress-Energy Tensor

The conformal transformation (30) affects both the geometry and the physical quantities relevant to the matter content of the gravitating source (the stress-energy tensor

of the perfect fluid), in a way that the dynamical description of gravity in the virtual case (the functional form of the Einstein field equations) remains invariant. In this respect, we consider that $\tilde{g}_{mn}$ is a solution of the *conformal Einstein field equations*

$$\tilde{\mathcal{R}}_{ml} - \frac{1}{2}\tilde{g}_{ml}\tilde{\mathcal{R}} = -\kappa\tilde{\mathcal{T}}_{ml} \tag{40}$$

where also

$$\kappa = \frac{8\pi G}{c^4}$$

By virtue of Eqs. (40), it is evident that the corresponding *conformal stress-energy tensor* ($\tilde{\mathcal{T}}_{ml}$) satisfies the conservation law $\tilde{\mathcal{T}}^{ml}_{;l} = 0$, where now the covariant derivative is taken with respect to the conformal metric, $\tilde{g}_{ml}$. In this case, we may deduce the exact form of the stress-energy tensor, responsible for the gravitational field of the virtual fluid, in terms of the corresponding tensor of the original source.

In four dimensions, the influence of a conformal transformation of the form (13) on the dynamical quantities describing the gravitational field (the Riemann tensor and its contractions) is (Hawking & Ellis 1973)

$$\tilde{\mathcal{R}}_{ml} = \mathcal{R}_{ml} - 2\Omega\left(\frac{1}{\Omega}\right)_{;\,ml} + \frac{1}{2}g_{ml}\frac{1}{\Omega^2}\Box\left(\Omega^2\right) \tag{41}$$

and

$$\tilde{\mathcal{R}} = \frac{1}{\Omega^2}\mathcal{R} + 6\frac{1}{\Omega^3}\Box\Omega \tag{42}$$

where the d' Alembert operator ($\Box$), with respect to the original metric, is given by

$$\Box\Omega = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^m}\left(\sqrt{-g}\, g^{mn}\,\Omega_{,\,n}\right).$$

With the aid of transformations (41) and (42), Eq. (40) reduces to

$$\begin{aligned}\tilde{\mathcal{T}}_{ml} &= \mathcal{T}_{ml} + \frac{2}{\kappa}\, g_{ml}\,\frac{\Box\Omega}{\Omega} - \frac{2}{\kappa}\,\frac{1}{\Omega}\,\Omega_{;\,ml} \\ &+ \frac{4}{\kappa}\,\frac{1}{\Omega^2}\,\Omega_{,\,m}\Omega_{,\,l} - \frac{1}{\kappa}\,g_{ml}\,\frac{1}{\Omega^2}\,g^{ns}\Omega_{,\,n}\Omega_{,\,s}\end{aligned} \tag{43}$$

and therefore, the trace of $\tilde{\mathcal{T}}_{ml}$ is written in the form

$$\tilde{\mathcal{T}} = \tilde{g}^{ml}\,\tilde{\mathcal{T}}_{ml} = \frac{1}{\Omega^2}\left[\mathcal{T} + \frac{3c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega}\right] \tag{44}$$

Clearly, Eqs. (43) and (44) hold for an arbitrary form of both $\tilde{\mathcal{T}}_{ml}$ and $\mathcal{T}_{ml}$. However, in connection to what previously stated, we may assume that the conformal stress-energy tensor can be written in the form of a perfect fluid, namely,

$$\tilde{\mathcal{T}}_{ml} = \left(\tilde{\mathcal{E}} + \tilde{p}\right)\tilde{u}_m\tilde{u}_l - \tilde{p}\,\tilde{g}_{ml} \tag{45}$$

for which we have

$$\tilde{\mathcal{T}} = \tilde{\mathcal{E}} - 3\,\tilde{p} \tag{46}$$

where, $\tilde{\mathcal{E}}$ and $\tilde{p}$ are the corresponding energy density and pressure. In this case, Eq. (45) results to

$$\tilde{\mathcal{E}} - 3\tilde{p} = \frac{1}{\Omega^2}\left[\mathcal{E} - 3p + \frac{3c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega}\right] \tag{47}$$

By virtue of Eq. (47) we perform the *ansatz* that, under the action of the conformal group (30), both $\mathcal{E}$ and p transform as follows

$$\tilde{\mathcal{E}} \;=\; \frac{1}{\Omega^2}\left[\mathcal{E} + a\,\frac{c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega}\right] \tag{48}$$

$$\tilde{p} \;=\; \frac{1}{\Omega^2}\left[p - (\frac{3-a}{3})\,\frac{c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega}\right] \tag{49}$$

where a is a numerical constant, the exact value of which can be determined in terms of the corresponding weak field-limit results [Eq. (39)] (see also Spyrou 1999). In analogy to Eq. (5), we assume that $\tilde{\mathcal{E}}$ may be decomposed as follows

$$\tilde{\mathcal{E}} = \tilde{\rho}c^2\,(1 + \frac{\tilde{\Pi}}{c^2}) \tag{50}$$

where $\tilde{\rho}\tilde{\Pi}$ corresponds to the specific internal energy density, related with the contractions or the expansions of the virtual fluid. Then, in the weak field-limit, Eq. (48) is reduced to

$$\begin{aligned}\tilde{\rho}\,(1 + \frac{\tilde{\Pi}}{c^2}) \;&=\; \rho\left[1 - \frac{1}{c^2}(\Pi + 2\frac{p}{\rho})\right] + \\ &+\; a\,\frac{1}{4\pi G}\left[1 - \frac{3}{c^2}(\Pi + \frac{p}{\rho})\right]\left(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[\Pi + \frac{p}{\rho}\right]\end{aligned} \tag{51}$$

which, in the stationary case, to lowest order in $\frac{1}{c^2}$ yields

$$\tilde{\rho} = \rho - a\,\frac{1}{4\pi G}\,\nabla^2\left(\Pi + \frac{p}{\rho}\right) \tag{52}$$

Now, by comparison of Eqs. (39) and (52), we obtain $a = -1$. Hence, Eqs. (48) and (49) are finally written in the form

$$\tilde{\mathcal{E}} \;=\; \frac{1}{\Omega^2}\left(\mathcal{E} - \frac{c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega}\right) \tag{53}$$

$$\tilde{p} \;=\; \frac{1}{\Omega^2}\left(p - \frac{c^4}{3\pi G}\,\frac{\Box\Omega}{\Omega}\right) \tag{54}$$

obviously satisfying Eqs. (47) - (49). As regards a self-gravitating perfect fluid, Eqs. (53) and (54) represent the transformation law of the corresponding energy density and pressure, under the action of the conformal group (30). It is worth noting that, according to Eq. (54), *the pressure of the virtual perfect fluid may not be constant (or vanish), although the motions in its interior are geodesics.* This result is a distinguishing feature of the present analysis, with respect to what we have encountered so far in relativistic astrophysics and cosmology, namely, that the isobaric flows are geodesics (see e.g. Weinberg 1972; Misner et al. 1973; Papapetrou 1974; Ryan & Shepley 1975; Narlikar 1983) and so, it justifies the use of the term *"virtual"*, in denoting the matter content responsible for the conformal metric.

In relation to the measurement data relevant to what we observe and determine observationally in the central regions of the AGNs, the theoretical result (53) has a clear physical interpretation: $\tilde{\mathcal{E}}$ is what we actually measure by assuming geodesic motions, while $\mathcal{E}$ is the real physical quantity associated to the fluid matter content existing in those regions. In this respect, the quantity

$$\rho_i = -\frac{c^2}{4\pi G}\,\frac{\Box\Omega}{\Omega} \tag{55}$$

may be identified as an **extra** inertial energy density, associated to the contributions from the *internal physical characteristics (internal motions, pressure, thermodynamic content)* of the original fluid, to the measured quantities. In fact, ρ_i describes the way the various physical characteristics of the source (beyond its rest-mass density) act as additional gravitational sources, thus affecting the geodesic motions (Spyrou & Dionysiou 1973; Spyrou 1997a). In this case, Eq. (53) is written in the form

$$\tilde{\mathcal{E}} = \frac{1}{\Omega^2}\left(\mathcal{E} + \rho_i c^2\right) \tag{56}$$

and the question arises as to whether the observationally measured energy $\tilde{\mathcal{E}}$ is being overestimated or not, with respect to the physical quantity $\mathcal{E}$.

According to Eqs. (55) and (56), the answer to the above question depends on the *sign* of $\Box\Omega$, which can be determined by taking into account the so called *weak energy condition.* It is known (Hawking & Penrose 1970) that, for any physical system (as it is the case for the original perfect fluid, described by $\mathcal{T}_{ml}$), the weak energy condition is valid, i.e. the locally measured energy is non-negative

$$\mathcal{T}_{ml}\, u^m u^l = \mathcal{E} \geq 0 \tag{57}$$

Admiting that the virtual fluid also represents a physical system, we demand that a condition similar to Eq. (57) is valid with respect to $\tilde{\mathcal{T}}_{ml}$ as well, namely,

$$\tilde{\mathcal{T}}_{ml}\, \tilde{u}^m \tilde{u}^l = \tilde{\mathcal{E}} \geq 0. \tag{58}$$

With the aid of Eq. (53), Eq. (58) is decomposed in terms of $\mathcal{E}$, as follows

$$\mathcal{E} \geq \frac{c^4}{4\pi G}\,\frac{\Box\Omega}{\Omega} \tag{59}$$

Now, by virtue of Eq. (57), the condition (59) is valid **for all** $\mathcal{E} \geq 0$, provided that

$$\Box\Omega \leq 0 \tag{60}$$

Therefore, we conclude that $\rho_i \geq 0$ and according to Eq. (56), the observationally determined energy ($\tilde{\mathcal{E}}$) **is being overestimated** as compared to the real, physical one ($\mathcal{E}$). The equality in Eq. (60) corresponds to the identity transformation case, representing an isobaric flow of the original fluid, for which, from Eqs. (20) and (22), we readily obtain $\Box\Omega = 0$.

6 Some Astrophysical Applications

Recently, it has been proposed that the centers of the AGNs contain large amounts of gas, in the form of warped disks or tori, which absorbs much of the non-thermal power generated in the central core (probably due to accretion phenomena) and reradiate it (Antonucci & Miller 1985; Guilbert & Rees 1988; Holt et al. 1992; Jaffe et al. 1993; Rydberg et al. 1993; Tacconi et al. 1994; Kohno et al. 1996; Greenhill & Gwinn 1997). Direct mapping of these gaseous structures can be possible with high spatial resolution spectroscopic observations of *water masers* (Urry & Padovani 1995). Indeed, VLBI imaging of water masers in NGC 4258 has provided a firm evidence for rotating molecular gas in a region less than 0.25 *pc* from the center of the galaxy (Greenhill et al. 1995; Miyoshi et al. 1995). Assuming Keplerian motion of the masering source, this imaging has allowed a quite accurate estimate of the central nuclear mass, which has been found to be a $3.6 \times 10^7\ M_\odot$ supermassive dark object, within 0.13 *pc*. However, combining these observations with the theoretical results obtained in the present article, it becomes evident that assuming Keplerian (geodesic) motions in the central region of the AGNs corresponds to work within the context of the virtual fluid. In this case, the *virtual results* can be extrapolated to the real, physical ones (based on hydrodynamic motions) through Eqs. (30), (53) and (54).

As a direct application, we will calculate the **mass-excess** m_i, arising in the determination of the central nuclear mass of the active galaxy NGC 4258, due to the assumption of Keplerian motion of the masering source. In fact, m_i represents the contribution of the fluid's internal characteristics (ρ_i) to the measured quantities. The corresponding calculations will be carried out in the weak field limit, since the various measurements based on observational data relevant to the central region of the AGNs, are usually derived by methods of Newtonian gravity. In this limit, the *superposition principle* is valid and therefore, each gravitating source (the black hole and/or the circumnuclear gas) is expected to contribute separately. For the sake of simplicity we shall assume that the nuclear region is *spherically symmetric*, so that all the physical quantities describing the galaxy's interior are functions of the radial coordinate r. We consider an idealized model, according to which the circumnuclear gas surrounding the central dark object can be represented by a stationary perfect fluid which undergoes adiabatic, hydrodynamical flow under the influence of its own

gravitational field, plus the gravitational field of the black hole (Novikov & Thorne 1973). In this case, dynamical equivalence between hydrodynamic flow and the Keplerian motion of a fluid's volume element implies that Eq. (35) holds, provided that the original gravitational potential U is now replaced by $U_{BH} + U_G$, where

$$U_{BH} = G \, \frac{M_{BH}}{r} \tag{61}$$

is the gravitational potential due to a (Schwarzschild) black hole of mass M_{BH}, and U_G is the corresponding potential of the self-gravitating gaseous matter in this region. Now, using Poisson's equation, Eq. (39) takes on the form

$$\tilde{\rho} = \rho + \frac{1}{4\pi G} \, \nabla^2 (\Pi + \frac{p}{\rho}) - \frac{1}{4\pi} \, M_{BH} \, \nabla^2 (\frac{1}{r}) \tag{62}$$

where $\nabla^2(\frac{1}{r}) = -4\pi\delta(r)$. The integration of Eq. (62) over a spatial volume $\mathcal{V}$ of linear dimension 0.25 pc, will give us the *virtual rest-mass* ($\tilde{m}$) of the central region in NGC 4258, namely,

$$\tilde{m} = (m + M_{BH}) + m_i \tag{63}$$

where m is the total rest-mass of the gas included in this region, while m_i represents the *relative error* introduced in the determination of the central mass by ignoring the contribution of the fluid's internal characteristics. Therefore, it becomes evident that the measured mass $\tilde{m}$ is being *overestimated* with respect to the real, physical quantity $m + M_{BH}$.

To determine the integral $m_i = \int_{\mathcal{V}} \rho_i d^3x$, we consider that the lower limit of integration corresponds to the radius of the *innermost stable circular geodesic*

$$r_0 = 3R_S = \frac{6GM_{BH}}{c^2} \tag{64}$$

We see that the value of r_0 is directly proportional to the mass of the central black hole. In general, the estimated masses of the galactic nuclear dark objects fall in the range $10^7 - 10^9 \, M_\odot$ (Kormendy & Richstone 1995). Accordingly, the corresponding values of r_0 range from 10^{-5} pc to 10^{-3} pc (in connection see Holt et al. 1992). In the present article we adopt a *mean value* of 10^{-4} pc. As regards to the upper limit of integration, we take $r_{max} = 0.25$ pc, which corresponds to the outer boundary of the masering annulus in NGC 4258 (Miyoshi et al. 1995). Moreover, by virtue of the adiabaticity condition (19), we obtain

$$\frac{1}{4\pi G} \, \nabla^2 (\Pi + \frac{p}{\rho}) = \rho_i = \frac{1}{4\pi G} \, \nabla \cdot (\frac{1}{\rho} \, \nabla p) \tag{65}$$

In the special case of adiabatic motions, the pressure is related to the rest-mass density through the *equation of state*

$$p = k \, \rho^\gamma \tag{66}$$

where $k > 0$ and $\gamma \geq 1$ are constants. Now Eq. (65) takes on the form

$$\rho_i = \frac{k\gamma}{4\pi G}\, \rho^{\gamma-2} \left[\nabla^2\rho + (\gamma-2)\, \frac{1}{\rho}\, (\nabla\rho{\cdot}\nabla\rho)\right] \tag{67}$$

In particular, we consider that the proper-mass density distribution can be described by a Plummer-type function (Maoz 1995) of the form

$$\rho = \rho_0\, (1+x^2)^{-\frac{n}{2}} \tag{68}$$

where $x = \frac{r}{r_0}$, while ρ_0 and n are possitive constants. In this case, Eq. (67) results to

$$\rho_i(x) = \frac{k\gamma n}{4\pi G}\, \frac{\rho_0^{\gamma-1}}{r_0^2}\, \frac{1}{(1+x^2)^{\frac{n}{2}(\gamma-1)+2}}\, \left\{[n(\gamma-1)+1]x^2 \,-\, 1\right\} \tag{69}$$

and the associated mass is written in the form

$$m_i = \frac{k\gamma n}{G}\, r_0\, \rho_0^{\gamma-1} \int_{x_0}^{x_{max}} \frac{x^2\, \{[n(\gamma-1)+1]x^2 \,-\, 1\}}{(1+x^2)^{\frac{n}{2}(\gamma-1)+2}}\, dx \tag{70}$$

where $x_0 = 1$ and $x_{max} = 0.25/10^{-4} = 2500$. The integral on the rhs of Eq. (70) can be expressed in terms of elementary functions only if

$$\frac{n}{2}\, (\gamma-1) \,+\, 2 = \ell \;:\, an\ integer \tag{71}$$

(Gradshteyn & Ryzhik 1965) and in fact, on physical grounds ($n > 0$ and $\gamma \geq 1$), we must have $\ell \geq 2$. The case $\ell = 2$ corresponds, for every value of n, to an *isothermal flow* of the perfect fluid, i.e. $\gamma = 1$. We shall consider this case separately. For $\ell \geq 2$, the general solution to Eq. (70) is written in the form (Gradshteyn & Ryzhik 1965)

$$\begin{aligned} m_i \;=\; & \frac{k\gamma n}{G}\, r_0\, \rho_0^{\gamma-1} \left[-\frac{2\ell-3}{2\ell-5}\, \frac{x^3}{(1+x^2)^{\ell-1}} - \frac{4(\ell-1)}{(2\ell-3)(2\ell-5)}\, \frac{x}{(1+x^2)^{\ell-1}} \right. \\ + \; & \frac{4(\ell-1)}{(2\ell-1)(2\ell-3)(2\ell-5)} \sum_{j=1}^{\ell-1} \frac{(2\ell-1)...(2\ell-2j+1)}{(\ell-1)...(\ell-j)}\, \frac{1}{2^j}\, \frac{x}{(1+x^2)^{\ell-j}} \\ + \; & \left. \frac{4(\ell-1)}{(2\ell-3)(2\ell-5)}\, \frac{(2\ell-3)!!}{(\ell-1)!}\, \frac{1}{2^{\ell-1}}\, tan^{-1}x \right]_{x_0}^{x_{max}} \end{aligned} \tag{72}$$

Clearly, Eq. (72) is too complicated to be useful in astrophysical applications. On the other hand, the isothermal case ($\ell = 2$) seems to be more appropriate. Indeed, recent results regarding the properties of *mass accretion* in NGC 4258 (Neufeld & Maloney 1995), indicate that the adiabatic flow of the the gaseous fluid is actually isothermal, with a constant *speed of sound* $c_s = 7\, km/sec$. In this case, $\gamma = 1$ and, therefore, Eq. (72) is written as

$$m_i = \frac{kn}{G}\, r_0\, \left[x\, (\frac{x^2+2}{x^2+1}) - 2tan^{-1}x \right]_{x_0}^{x_{max}} \tag{73}$$

which, by virtue of Eq. (64), results to

$$m_i = (kn) \cdot 1.66 \times 10^{-17} \, M_{BH} \tag{74}$$

However, in the case of isothermal flow, we furthermore have

$$k = c_s^2 \tag{75}$$

and, therefore, as regards NGC 4258, adopting the typical values $c_s = 7 \; km/sec$ (Neufeld & Maloney 1995) and $n = 5$ (Miyoshi et al. 1995) we finally obtain

$$m_i \sim 5 \times 10^{-5} \, M_{BH} \tag{76}$$

This very small relative error, is consistent with observations and in fact, provides a theoretical explanation for the almost perfect Keplerian rotation-curve observed for the gas, in the central region of this particular galaxy (Greenhill et al. 1995; Maoz 1995; Miyoshi et al. 1995). Nevertheless, had we applied the same model in NGC 1068, where the outer radius of the masering source is $r_{max} = 1 \; pc$ (Greenhill & Gwinn 1997), we would have obtained

$$m_i \sim 2 \times 10^{-4} \, M_{BH} \tag{77}$$

while, in the extreme case of NGC 4261, where (non-masering) circumnuclear gas and dust appears to extend up to $r_{max} \sim 150 \; pc$ from the center (Jaffe et al. 1993), we would have accordingly obtained

$$m_i \sim 3 \times 10^{-2} \, M_{BH} \tag{78}$$

Therefore, m_i **is not always negligible compared to the mass of the central dark object** and it can account from a few hundredths of thousanths to several hundredths of M_{BH}, depending on the linear dimensions of the circumnuclear gas observed in the AGNs.

On the other hand, the assumption of Keplerian motions in the cental region of the AGNs is compatible with the existence of circumnuclear gas around a black hole, only if the central dark object contains at least 98% of the galactic nuclear mass (Greenhill et al. 1995). In fact, calculations based on Newtonian dynamics indicate that the total rest-mass of the gas is $m \sim \frac{1}{50} M_{BH}$ (Maoz 1995). In this case, we may also calculate the ratio $\frac{m_i}{m}$ for the particular galaxies considered above. Accordingly, for NGC 4258, we find

$$m_i = 2.5 \times 10^{-3} \, m \tag{79}$$

while for NGC 1068 and NGC 4261 we obtain $m_i \sim 10^{-2} \, m$ and $m_i \sim 1.5 \, m$, respectively. Therefore, m_i **can be comparable to or even larger than the total rest-mass of the circumnuclear gas involved**.

At the outcome, we have to point out that the above idealized scheme is rather naive in realistic situations. In fact, it is expected that in the central regions of the

AGNs the flow motion is not adiabatic at all. Significant energy losses may occur, due to thermal bremsstrahlung, synchrotron radiation and/or radiative transfer (Novikov & Thorne 1973), leading to a loss of angular momentum, so that, ultimately, the matter surrounding the central dark object will end up into the hole (Shakura & Sunyaev 1973). It has been found (Neufeld & Maloney 1995) that, for a mass accretion rate of the order $\dot{m} = 7 \times 10^{-5}\ M_{\odot}\ yr^{-1}$, material located at a distance 0.25 pc from the center will fall into the hole after $4 \times 10^6\ yrs$. Therefore, the case of a *viscous accretion disk* seems to be more appropriate than the spherically symmetric perfect fluid distribution, but it will not be considered here.

7 Discussion and Conclusions

Although recent developments in galactic dynamics indicate that the center of the AGNs contains large amounts of gas (Urri & Padovani 1995), the observational determination of the central nuclear masses is based on the assumption of pure geodesic motions (Kormendy & Richstone 1995). Stimulated by this contradiction, we have examined the conditions under which the hydrodynamic flow in the interior of a self-gravitating perfect fluid can become dynamically equivalent to geodesic motion in the same source. By definition, dynamical equivalence rests on the functional similarity between the corresponding differential equations of motion. In this case, the spaces of the solutions of these two kinds of motion are *isomorphic*. In other words, given a solution to a problem of hydrodynamic flow in a gravitating perfect fluid, one can always construct a solution formally equivalent to the problem of geodesic motion in the same source and vice versa.

As regards a self-gravitating perfect fluid with metric g_{ml}, dynamical equivalence between the two kinds of motion can be achieved in the case of *adiabatic flow*. Then, we actually transfer the problem to a *virtual* fluid, with metric $\tilde{g}_{ml}$, in which the motions are geodesics. In this case, the two metrics are connected by means of a *conformal transformation* [Eq. (13)], with the identity element representing an *isobaric flow* of the original fluid.

In general, to determine the functional form of the *conformal factor* $\Omega(x^i)$, involves the solution of the *compatibility equation* (24). In the present article we have considered the particular solution $\Phi = constant$. The resulting conformal transformation affects both the geometry and the stress-energy tensor ($\mathcal{T}_{ml}$) of the original fluid, in a way that the functional form of the Einstein field equations remains invariant. In this respect, we have also determined the form of the *virtual stress-energy tensor* ($\tilde{\mathcal{T}}_{ml}$) in terms of the original one. In comparison to the measurements relevant to the observational data from the central region of the AGNs, these two quantities have a clear physical interpretation: $\tilde{\mathcal{T}}_{ml}$ is what we actually measure by assuming geodesic motions, while $\mathcal{T}_{ml}$ corresponds to the "real" stress-energy tensor associated to the fluid matter content in those regions. In this case, with the aid of the weak energy condition, we have found that the observationally determined quantity ($\tilde{\mathcal{T}}_{ml}$) is being

overestimated with respect to $\mathcal{T}_{ml}$.

Admiting an idealized model in which a spherically symmetric perfect fluid surrounds a massive black hole, we have applied the previous results in the determination of the central nuclear mass, as regards some particular AGNs. In the Newtonian limit, the corresponding results indicate that the *mass-excess* (m_i) arising in the observational determination of the central masses by ignoring the contribution of the fluid's internal characteristics, is not always negligible compared to the mass of the black hole. In fact, it can account from a few hundredths of thousanths (in NGC 4258) to several hundredths (in NGC 4261) of M_{BH}, depending on the linear dimension of the circumnuclear gas. On the other hand, we have shown that m_i can be either comparable or even larger than the total rest-mass of the fluid involved.

However, the assumption of a (spherically symmetric) perfect fluid can be rather naive in realistic situations. In this case, the *viscous hydromagnetic flow* in the interior of a *warped accretion disk* should be more appropriate, but the corresponding equations are too complicated for astrophysical applications. Furthermore, it can be verified that in the case of a *magnetized self-gravitating perfect fluid*, the determination of the corresponding conformal transformation involves the solution of a compatibility condition of the form

$$h^{mn} \frac{\Omega_{,\,n}}{\Omega} = - \frac{e}{m_e c} \mathcal{F}^m_n \, u^n \tag{80}$$

where, m_e is the mass of the charge e and $\mathcal{F}^m_n$ is the antisymmetric tensor of the electromagnetic field involved. Clearly, Eq. (80) is far more complicated than Eq. (24).

Finally, one cannot help asking whether dynamical equivalence between the two kinds of motion implies physical equivalence, in general. Clearly, the geodesic motion of a test-particle is an approximate description with certain limits of validity. The test-particle does not react back to modify the original gravitational field. To decide on how realistic is that concept, is an old and very interesting problem. In this respect, the conformal transformation (30) provides a direct way of linking the laws of motion of an idealized point-particle with those determining the hydrodynamic flow of a realistic volume element in the interior of a continuous source. In the latter case, the problem of *backreaction* is directly involved in the structure of the corresponding stress-energy tensor. To which extend is this procedure applicable to any form of $\mathcal{T}_{ml}$, remains an open problem.

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